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REENTRY TRACKING MODELS IN RADAR POLAR COORDINATES AND STATISTICAL TESTING PROCEDURES FOR MODEL SIMPLIFICATION

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ABSTRACT

The following report describes (a) an outline of the Kalman-based tracking procedure for reentry trajectories, (b) a detailed derivation of a mathematical model to implement this procedure in radar polar coordinates, and (c) methods for checking, via simulation, the degree of degradation introduced by model simplifications and other changes.

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I. RECURSIVE TRACKING

1. Introduction

The following report describes the derivation and implementation of methods and models for simulating the reentry tracking of an object in radar-centered polar coordinates and the statistical comparison of various tracking algorithms. This work was performed for Lincoln Laboratory, M.I.T., under the direction of Dr. Richard Wishner, Assistant Group Leader, Group 42, in support of a program for advancing the techniques of reentry tracking from radar data.

The target tracking procedure considered here is a relatively straightforward application of Kalman's recursive estimation technique[4]. This application was reported upon previously by Gruber[2], forming the basis for subsequent computer simulation models. The early models were based on a radar-centered rectangular coordinate system and were programmed and investigated by Bertolini[3].

A radar-centered polar coordinate system has the relative advantage of leading to a simple linear model for observation geometry at the expense of a more complicated dynamical model. It has been suspected that the linear observation model represents a definite advantage in tracking stability and accuracy, especially just after target acquisition. The present work is a contribution to the polar coordinate studies.

Dynamical equations for a trajectory in radar polar coordinates have been given by Catalano[1]. We rederive these equations here in a matrix format and continue with a development of the partial derivative matrices for the spherical earth case, which are necessary for the tracking algorithms.

In summary, the report contains (a) an outline of the Kalman-based tracking procedure for reentry trajectories, (b) a detailed derivation of a mathematical model to implement this procedure in radar polar coordinates, (c) methods for checking, via simulation, the degree of degradation introduced by model simplifications and other changes.

An Appendix summarizes the rather extensive notation conventions and symbology used in this report.

2. Tracking Models

The tracking procedure, as we noted above, is based on Kalman's method of recursive estimation. This estimation technique is ideally suited to the present problem because

a. It is a general method. Many different variations of reentry dynamics and observational procedures can be handled within a common framework. Thus design experimentation is facilitated.

b. It is real-time estimation method. The utilization of tracking data in forming the track estimate is sequential in time. Since we are interested in applying the results of our studies to real-time systems, we are obliged to confine ourselves to such methods.

c. It provides optimal estimates where the model equations are linear, and approximate, optimal estimates in the nonlinear case.

In view of these properties of recursive estimation it is not surprising that most of the tracking methods formulated in the past for real-time applications turn out to be special cases of the general Kalman formulation. While it is possible to conceive of tracking schemes which do not fit such a formulation, the investigation of each such scheme would be a new and unique undertaking. We believe that at present the best service is performed by investigating the Kalman class of tracking models and will confine our discussion accordingly.

In order to implement a particular model, we must postulate or derive two sets of equations. The first set, a system of first order differential equations, describes the dynamics of the reentry body and its motion along the trajectory. The second set describes the geometry of each measurement. Since it is clear that the same physical trajectory can be treated in different coordinate frameworks and that varying levels of detail can be

incorporated in the dynamics, a class of models arises, each treating the same problem.

The dynamical equations may be written in vector form

(1.1) $$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, t)$$

where $\underline{x}$ components are the dynamical or so-called state variables and t is time. Similarly, the observation geometry may be written in the form

(1.2) $$\underline{z} = \underline{h}(\underline{x}, t) + \underline{w}(t)$$

where $\underline{z}$ is the measurement vector at time t and $\underline{w}$ is an additive noise vector. This noise vector is required to be statistically uncorrelated for different t and has the assigned covariance matrix

(1.3) $$W(t) = \text{cov}(\underline{w}(t)) .$$

Without loss of generality $\underline{w}(t)$ can be assumed to have zero mean for each t.

The dynamical equations may be generalized to account for random disturbances by including random driving terms on the right side.

(1.4) $$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, t) + \underline{w}_*(t)$$

In this more general form, $\underline{w}_*(t)$ is assumed to be a zero mean white noise process with given covariance matrix, $W_*(t)$. In the following discussion, the simpler dynamical equations (1.1) will be used. Then the modifications necessary to include the effects of a random driving term, $\underline{w}_*(t)$, will be noted.

Owing to the nonlinear nature of the above equations ($\underline{f}$ and $\underline{h}$ are nonlinear functions of $\underline{x}$), it is inevitable that statistical calculations be based on a linearization of (1.1) and (1.2). Thus, to compute tracking errors and their statistics, we assume that the reentry object exhibits

only small deviations from a nominal trajectory and from (1.1) derive an equation for the propagation of such deviations. Let us therefore consider a specific trajectory, $\underline{x}_0(t)$, satisfying (1.1), as a reference trajectory

$$\frac{d\underline{x}_0}{dt} = f(\underline{x}_0, t) \tag{1.5}$$

The actual trajectory will differ from $\underline{x}_0$ by an amount $\delta\underline{x}$, assumed small.

$$\delta\underline{x} = \underline{x} - x_0 \tag{1.6}$$

From (1.1) we have

$$\frac{d\underline{x}_0}{dt} + \frac{d}{dt}\delta\underline{x} = \underline{f}(\underline{x}_0 + \delta\underline{x}, t) .$$

Expanding terms on the right in a power series to the first order in $\delta\underline{x}$ and using (1.5), we find a matrix equation for the dynamics of $\delta\underline{x}$.

$$\frac{d}{dt}\delta x = J(\underline{x}_0, t)\,\delta\underline{x} \tag{1.7}$$

Here J is the jacobian matrix of $\underline{f}$ with components

$$J_{ij} = \frac{\partial f_i}{\partial x_j} \tag{1.8}$$

evaluated on the reference trajectory.

Similarly, we use as inputs to a statistical calculation, deviations of the actual observation vector, $\underline{z}$, from the ideal observations, $\underline{z}_0$, that would be taken on the assumed reference trajectory with no noise. That is,

$$\delta\underline{z} = \underline{z} - \underline{z}_0 \tag{1.9}$$

where

(1.10) $\underline{z}_0 = \underline{h}(\underline{x}_0, t)$.

Again assuming that $\delta \underline{z}$ and $\delta \underline{x}$ are small, making a first order approximation in (1.2) and using (1.6) and (1.10), we find from

$$\underline{z}_0 + \delta \underline{z} = \underline{h}(\underline{x}_0 + \delta \underline{x}, t) + \underline{w}(t)$$

that

(1.11) $\delta \underline{z} = H(\underline{x}_0, t)\, \delta \underline{x} + \underline{w}(t)$.

Here H is a matrix of first partials of $\underline{h}$ with components

(1.12) $H_{kj} = \dfrac{\partial h_k}{\partial x_j}$

evaluated on the reference trajectory.

3. Tracking Recursions

In outlining the method of tracking by the recursive technique with the above model, we assume that data is received at discrete times t_n ($n = 1, 2, 3, \ldots$) and introduce the following quantities and notation.

- $\underline{z}(n)$ - the data received at time n.
- $\underline{x}(n)$ - the value of the state vector at time n.
- $\hat{\underline{x}}(n|n)$ - an estimate of $\underline{x}(n)$ given the data up to and including $\underline{z}(n)$.
- $\hat{\underline{x}}(n|n-1)$ - an estimate of $\underline{x}(n)$ given data up to and including $\underline{z}(n-1)$.
- $\Sigma(n|n-1)$ - the covariance matrix of errors in $\hat{\underline{x}}(n|n-1)$.
- $H(n)$ - partial derivative matrix (1.12) at t_n
- $W(n)$ - measurement noise covariance matrix (1.3) at t_n.

In what follows, ^{t} denotes matrix transposition.

We begin by assuming that we are in possession of $\hat{\underline{x}}(n-1|n-1)$ and $\Sigma(n-1|n-1)$. Then the recursive method consists of the following steps:

a. Extrpolate $\hat{\underline{x}}(n-1|n-1)$ forward to time t_n thus obtaining $\hat{\underline{x}}(n|n-1)$. The method of extrapolation is to use $\hat{\underline{x}}(n-1|n-1)$ as the initial vector in the dynamical equations (1.1) and to numerically integrate these equations up to time t_{n+1}.

(1.13) $$\frac{d\hat{\underline{x}}}{dt} = f(\hat{x}, t)$$

The solution at time t_{n+1} is $\hat{x}(n|n-1)$.

b. Extrapolate $\Sigma(n-1|n-1)$ forward to time t_n thus obtaining $\Sigma(n|n-1)$. The method here is to adapt the linearized error propagation equation (1.7). The corresponding propagation equation for covariances has the form

(1.14) $$\frac{d\Sigma}{dt} = J \Sigma J^t$$

where, ideally, J is the jacobian (1.8) evaluated on the true trajectory. However, for a real estimation problem, the true trajectory is not known. Hence, we evaluate J instead on the estimated trajectory between t_{n-1} and t_n determined by the extrapolation process of a. Then $\Sigma(n-1|n-1)$ is used as an initial matrix value in (1.14) and this matrix equation in numerically integrated forward to provide a solution $\Sigma(n|n-1)$ at time t_n.

c. At time t_n, new data $\underline{z}(n)$ are received; $\Sigma(n|n)$ and $\hat{\underline{x}}(n|n)$ are generated from $\Sigma(n|n-1)$ and $\hat{\underline{x}}(n|n-1)$ by the Kalman formulas

(1.15) $$\Sigma(n|n) = \Sigma(n|n-1) - \Sigma(n|n-1) H^t(n) [H(n) \Sigma n|n-1) H^t(n) + W(n)]^{-1} H(n) \Sigma(n|n-1)$$

$$\hat{\underline{x}}(n|n) = \hat{\underline{x}}(n|n-1) + \Sigma(n|n-1)\, H^t(n)\, [H(n)\, \Sigma(n|n-1)\, H^t(n) + W(n)]^{-1} [\underline{z}(n) - H(n)\, \hat{\underline{x}}(n|n-1)] \tag{1.16}$$

The cycle of calculations then repeats, returning to step a. The successive values of $\hat{x}(n|n)$ thus generated are the tracking estimates. Of course, an independent method is required to supply the initial starting values $\hat{x}(0|0)$ and $\hat{\Sigma}(0|0)$.

The above procedure may be generalized slightly to account for the presence of random driving terms in the dynamical equations as in (1.4). In most cases such terms in the model are added to account for unknown effects and to hedge the statistical estimation process. For such a purpose it is equally appropriate to add the random term only at the discrete observation times. Thus instead of analyzing the effect of a continuous excitation $w_*(t)$, we simply replace it with a discrete one $w_*(n)$ with an appropriate covariance $W_*(n)$. Only the above step b. is modified to account for this complication. When $\Sigma(n-1|n-1)$ is extrapolated forward, the matrix $W_*(n-1)$ is added to the final extrapolated value in order to obtain $\Sigma(n|n-1)$.

II. RADAR POLAR COORDINATE MODEL ELLIPSOIDAL EARTH

1. Earth-Station Geometry

It should be clear that if the same physical problem is described in different coordinate systems the functions $\underline{f}$, $\underline{h}$, and J, H of the model will differ in a corresponding way. Since the effects of the linear approximations on the accuracy and stability of subsequent statistical calculations depend on the precise nonlinear forms of $\underline{f}$, $\underline{h}$, it is of some interest to explore various coordinate descriptions for their influence on these numerical properties. At the same time, the level of complexity included in the dynamical model may be varied to account for the various physical influences and the properties of the reentry body.

In this report, we shall describe in detail one such coordinate system and level of model complexity. The chosen coordinate system is earth fixed, polar, and centered at an observation site. Observations are assumed to be made by a single radar and consist of measurements of range R, azimuth A, elevation E, and their rates (or a suitable subset of these). By choosing these very same variables as dynamical variables, we insure that the functions $\underline{h}$ are exactly linear and that no approximation need be made on deriving H from $\underline{h}$. On the other hand, the dynamical laws (1.1) are complicated by this choice.

The dynamical model assumes a rotating earth of ellipsoidal figure, gravity terms including the first zonal harmonic (J_2) and air drag on a point mass non-lifting reentry body. This model is essentially that of Catalano; the derivations given here are somewhat more detailed, and the results are adapted to the needs of programming and statistical investigations.

In the following derivation vectors, which are always underlined, are understood to be expressed in column matrix form in a particular orthogonal coordinate system. Thus, the physical vector from the radar to the tracked object, the "$\underline{R}$" vector, is $\underline{R}$ in the unprimed coordinate system, $\underline{R}'$ in the primed system, and $\underline{R}''$ in the double primed system. These are

all different matrices, although representing the same physical vector, and have a common length R, the range measurement. When the physical vector is discussed, the particular representation $\underline{R}$ will be used for convenience. These conventions will be followed in all vector notation, except that the physical unit vectors of the unprimed system ($\underline{u}_1$, $\underline{u}_2$, $\underline{u}_3$), of the primed system ($\underline{u}_1'$, $\underline{u}_2'$, $\underline{u}_3'$), and of the double primed system ($\underline{u}_1''$, $\underline{u}_2''$, $\underline{u}_3''$) represent truly different vectors, not different representations of the same vectors. They bear the prime notation to distinguish the coordinate set.

Consider then an oblate earth of equatorial radius a_e and eccentricity e, rotating at the siderial rate ω. Let (u_1, u_2, u_3) be an inertial orthogonal coordinate system where u_1, u_2 are in the equatorial plane and u_3 points toward the north pole. Assume a radar station at distance a from earth's center, at geocentric colatitude φ and lying at longitude θ with respect to the u_1 axis. Figure 1 depicts these relations.

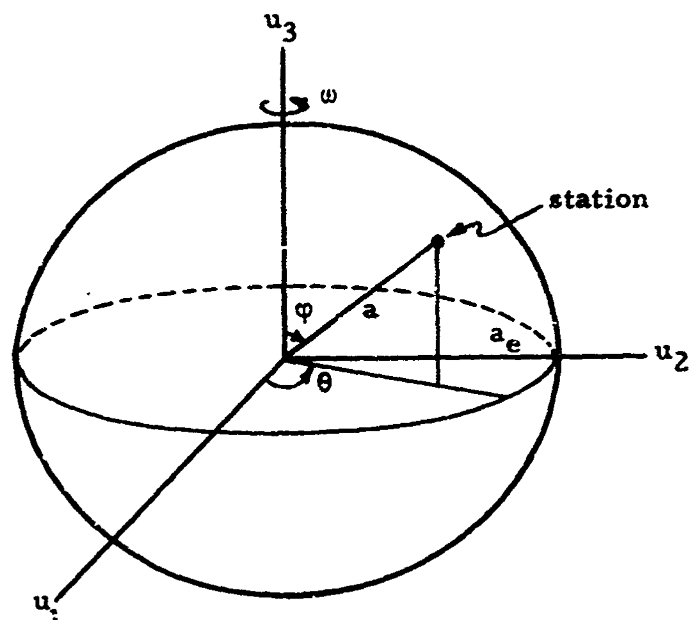

Figure 1.

Since the station is fixed with respect to the earth, we have

$$\dot{\theta} = \omega \qquad \text{or} \qquad \theta = \theta_0 + \omega t \quad .$$

The polar radius of the earth is $a_e\sqrt{1 - e^2}$.

Now consider a cross-section of the earth in the plane of the station, Figure 2. h_s is the altitude of the station, μ its geodetic latitude, and μ' its geocentric latitude. u_s is the equatorial coordinate in the station plane.

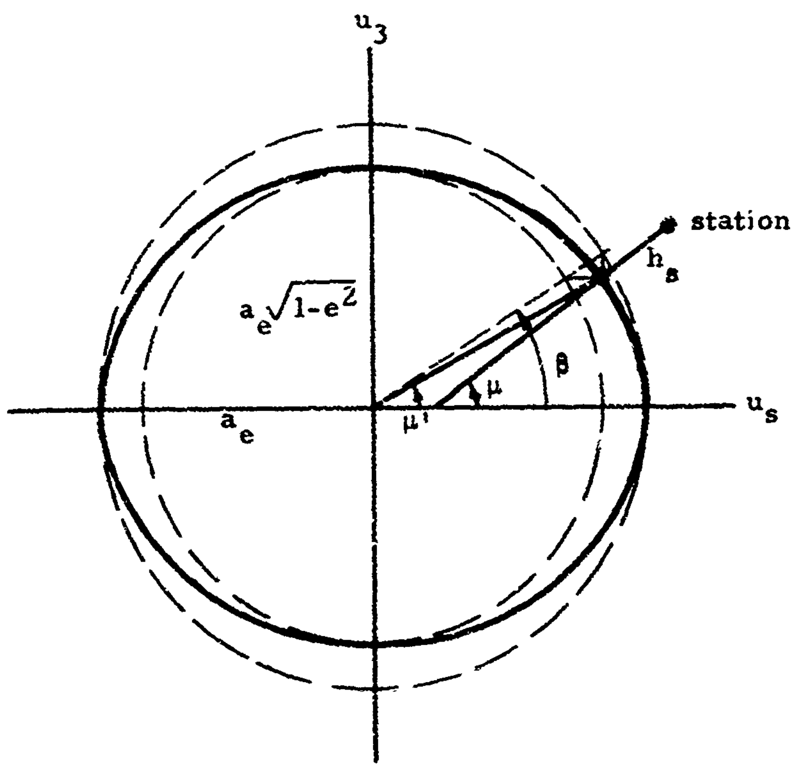

Figure 2.

Also shown in Figure 2 are the two construction circles of radii a_e and $a_e\sqrt{1 - e^2}$ for the parametric representation of the elliptical figure with the aid of the angle β. Thus, the ground projection of the station has coordinates

$$(2.1) \qquad \begin{aligned} u_s &= a_e \cos\beta \\ u_3 &= a_e\sqrt{1 - e^2}\sin\beta \end{aligned}$$

If β is varied by a small amount, u_s and u_3 change accordingly

$$du_s = -a_e \sin\beta \quad d\beta$$

$$du_3 = a_e\sqrt{1-e^2}\cos\beta \quad d\beta \quad .$$

Therefore, the (u_s, u_3) vector with components $(-\sin\beta, \sqrt{1-e^2}\cos\beta)$ will be a north-pointing tangent vector on the surface at the ground projection of the station. A perpendicular vector, $(\sqrt{1-e^2}\cos\beta, \ \sin\beta)$, will be an outward normal at the ground point. Thus

$$\cos\mu = k\cdot\sqrt{1-e^2}\cos\beta$$

$$\sin\mu = k\cdot\sin\beta$$

or conversely

$$\cos\beta = \frac{1}{k}\frac{\cos\mu}{\sqrt{1-e^2}}$$

$$\sin\beta = \frac{1}{k}\sin\mu \ ,$$

where k is chosen so that the necessary identity $\cos^2\beta + \sin^2\beta = 1$ is satisfied. We obtain

$$\cos\beta = \frac{\cos\mu}{\sqrt{1-e^2\sin^2\mu}}$$

$$\sin\beta = \frac{\sqrt{1-e^2}\sin\mu}{\sqrt{1-e^2\sin^2\mu}} \quad .$$

Using (2.1), we obtain the ground point coordinates in terms of the geodetic latitude. Finally, adding an increment for station height, we obtain the station coordinates.

$$(2.2) \qquad u_s = \frac{a_e \cos\mu}{\sqrt{1 - e^2 \sin^2\mu}} + h_s \cos\mu$$

$$u_3 = \frac{a_e (1 - e^2) \sin\mu}{\sqrt{1 - e^2 \sin^2\mu}} + h_s \sin\mu$$

For convenience define

$$(2.3) \qquad a_1 = \frac{a_e}{\sqrt{1 - e^2 \sin^2\mu}} + h_s$$

$$a_2 = \frac{a_e (1 - e^2)}{\sqrt{1 - e^2 \sin^2\mu}} + h_s$$

then using (2.2), the station vector (coordinates) in the (u_1, u_2, u_3) system can be written

$$(2.4) \qquad \underline{a} = \begin{bmatrix} a_1 \cos\mu \cos\theta \\ a_1 \cos\mu \sin\theta \\ a_2 \sin\mu \end{bmatrix}$$

Note that a, the magnitude of $\underline{a}$, is given by

$$(2.5) \qquad a^2 = a_1^2 \cos^2\mu + a_2^2 \sin^2\mu$$

Noting that $\dot{\theta} = \omega$, we find the rates of change of $\underline{a}$ in the inertial system to be

$$(2.6) \qquad \underline{\dot{a}} = a_1 \omega \cos\mu \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

$$\underline{\ddot{a}} = - a_1 \omega^2 \cos\mu \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

2. Observation Geometry

The station observations are made from position $\underline{a}$ but in an earth fixed system with origin at $\underline{a}$. Define an orthogonal system at $\underline{a}$ and make observations as shown in Figure 3.

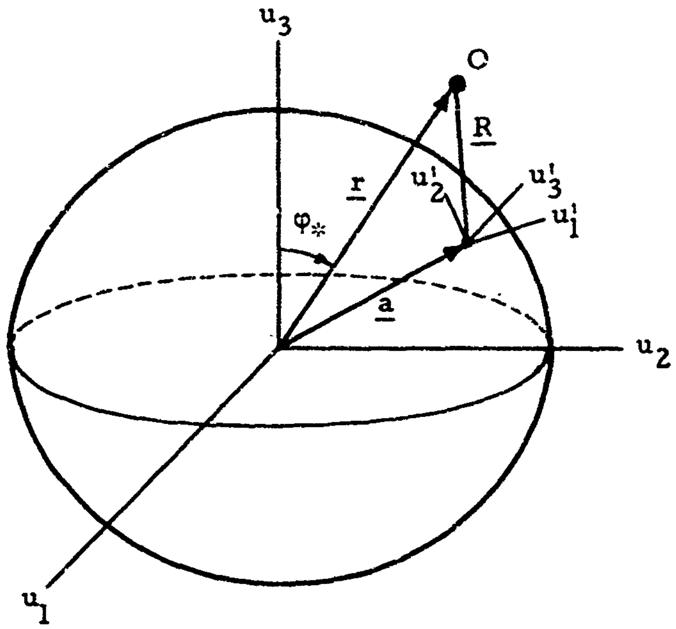

Figure 3.

The observed object is at point O at position vector $\underline{r}$; the observation vector is $\underline{R}$. In the local system (u_1', u_2', u_3'), u_3' is perpendicular to the earth's surface (at the stations ground point), u_1' points tangentially east, and u_2' tangentially north. $\underline{R}$ may also be described in terms of range, azimuth, and elevation relative to the u' system as shown in Figure 4.

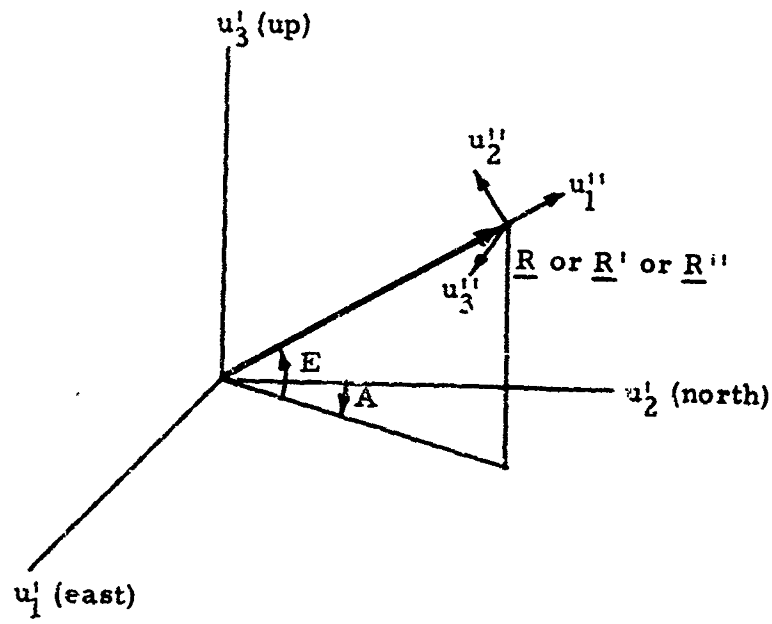

Figure 4.

A third set of orthogonal axes may then be constructed with u_1'' along with the R direction, u_3'' in the A direction parallel to the ground plane, and u_2'' in the E direction to form a right-handed system. By projecting the u'' unit vectors on to the u' axes, we find the vector relations,

(2.7)
$$\begin{aligned}
\underline{u}_1'' &= \cos E \sin A\, \underline{u}_1' + \cos E \cos A\, \underline{u}_2' + \sin E\, \underline{u}_3' \\
\underline{u}_2'' &= -\sin E \sin A\, \underline{u}_1' - \sin E \cos A\, \underline{u}_2' + \cos E\, \underline{u}_3' \\
\underline{u}_3'' &= \cos A\, \underline{u}_1' - \sin A\, \underline{u}_2'
\end{aligned}$$

These provide an orthogonal transformation, T_*, for transforming vector components from the u'' to the u' system,

(2.8)
$$T_* = \begin{bmatrix} \cos E \sin A & -\sin E \sin A & \cos A \\ \cos E \cos A & -\sin E \cos A & -\sin A \\ \sin E & \cos E & 0 \end{bmatrix}.$$

For example, the physical vector $\underline{R}$ has the components (R, 0, 0) in the u'' system. In the u' system, the coordinates are

$$(2.9) \qquad \underline{R}' = T_* \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R \cos E \sin A \\ R \cos E \cos A \\ R \sin E \end{bmatrix}.$$

But the $\underline{R}$ vector components are also related to $\underline{R}'$ by an orthogonal matrix transformation.

$$(2.10) \qquad \underline{R} = T\,\underline{R}'$$

From the geometry implied by Figures 1 and 3, we can immediately project unit vectors $\underline{u}_1'$, $\underline{u}_2'$, $\underline{u}_3'$ into the inertial system and represent them as linear combinations of unit vectors $\underline{u}_1$, $\underline{u}_2$, $\underline{u}_3$ in this system. We obtain the vector relations,

$$\begin{aligned} \underline{u}_1' &= -\sin\theta\,\underline{u}_1 + \cos\theta\,\underline{u}_2 \\ \underline{u}_2' &= -\sin\mu\cos\theta\,\underline{u}_1 - \sin\mu\sin\theta\,\underline{u}_2 + \cos\mu\,\underline{u}_3 \\ \underline{u}_3' &= \cos\mu\cos\theta\,\underline{u}_1 + \cos\mu\sin\theta\,\underline{u}_2 + \sin\mu\,\underline{u}_3 \quad . \end{aligned}$$

These relations immediately yield the required transformation,

$$(2.11) \qquad T = \begin{bmatrix} -\sin\theta & -\sin\mu\cos\theta & \cos\mu\cos\theta \\ \cos\theta & -\sin\mu\sin\theta & \cos\mu\sin\theta \\ 0 & \cos\mu & \sin\mu \end{bmatrix}.$$

3. Equations of Motion

From Figure 3, we see that

$$(2.12) \qquad \underline{r} = \underline{a} + \underline{R}$$

Then using (2.13) and differentiating, we find the succession of relations

$$(2.13)\qquad \begin{aligned} \underline{r} &= \underline{a} + T\,\underline{R}' \\ \underline{\dot{r}} &= \underline{\dot{a}} + T\,\underline{\dot{R}}' + \dot{T}\,\underline{R}' \\ \underline{\ddot{r}} &= \underline{\ddot{a}} + T\,\underline{\ddot{R}}' + 2\dot{T}\,\underline{\dot{R}}' + \ddot{T}\,\underline{R}' \ . \end{aligned}$$

By Newton's law ($\underline{r}$ is in an inertial system) $\underline{\ddot{r}}$ may be equated to the specific force vector applied to the reentry body.

$$(2.14)\qquad \underline{\ddot{a}} + T\,\underline{\ddot{R}}' + 2\dot{T}\,\underline{\dot{R}}' + \ddot{T}\,\underline{R}' = \underline{F}/m$$

Now by direct differentions, the left side of (2.14) is evaluated using (2.6), (2.8), (2.9), and (2.11). First the T derivatives may be evaluated in the following form.

$$(2.15)\qquad \dot{T} = \begin{bmatrix} -\omega\cos\theta & \omega\sin\mu\sin\theta & -\omega\cos\mu\sin\theta \\ -\omega\sin\theta & -\omega\sin\mu\cos\theta & \omega\cos\mu\cos\theta \\ 0 & 0 & 0 \end{bmatrix}$$

$$= T\omega \begin{bmatrix} 0 & -\sin\mu & \cos\mu \\ \sin\mu & 0 & 0 \\ -\cos\mu & 0 & 0 \end{bmatrix} .$$

$$(2.16)\qquad \ddot{T} = \dot{T}\omega \begin{bmatrix} 0 & -\sin\mu & \cos\mu \\ \sin\mu & 0 & 0 \\ -\cos\mu & 0 & 0 \end{bmatrix}$$

$$= -T\omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\mu & -\sin\mu\cos\mu \\ 0 & -\sin\mu\cos\mu & \cos^2\mu \end{bmatrix} .$$

In order to evaluate $\underline{R}'$ derivatives, we first calculate $\dot{T}_*$ in the form

$$(2.17)\quad \dot{T}_* = \begin{bmatrix} -\dot{E}\sin E\sin A+\dot{A}\cos E\cos A & -\dot{E}\cos E\sin A-\dot{A}\sin E\cos A & -\dot{A}\sin A \\ -\dot{E}\sin E\cos A-\dot{A}\cos E\sin A & -\dot{E}\cos E\cos A+\dot{A}\sin E\sin A & -\dot{A}\cos A \\ \dot{E}\cos E & -\dot{E}\sin E & 0 \end{bmatrix}$$

$$= T_* \begin{bmatrix} 0 & -\dot{E} & -\dot{A}\cos E \\ \dot{E} & 0 & \dot{A}\sin E \\ \dot{A}\cos E & -\dot{A}\sin E & 0 \end{bmatrix} .$$

Then

$$(2.18)\quad \dot{\underline{R}}' = T_* \begin{bmatrix} \dot{R} \\ 0 \\ 0 \end{bmatrix} + \dot{T}_* \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

$$= T_* \begin{bmatrix} \dot{R} \\ R\dot{E} \\ R\dot{A}\cos E \end{bmatrix} .$$

and

$$(2.19)\quad \ddot{\underline{R}}' = T_* \begin{bmatrix} \ddot{R} \\ \dot{R}\dot{E} + R\ddot{E} \\ \dot{R}\dot{A}\cos E + R\ddot{A}\cos E - R\dot{A}\dot{E}\sin E \end{bmatrix} + \dot{T}_* \begin{bmatrix} \dot{R} \\ R\dot{E} \\ R\dot{A}\cos E \end{bmatrix}$$

$$= T_* \begin{bmatrix} \ddot{R} - R\dot{E}^2 - R\dot{A}^2\cos^2 E \\ R\ddot{E} + 2\dot{R}\dot{E} + R\dot{A}^2\sin E\cos E \\ R\ddot{A}\cos E + 2\dot{R}\dot{A}\cos E - 2R\dot{A}\dot{E}\sin E \end{bmatrix} .$$

By directly substituting and rearranging these results, the left side of (2.14) takes the form

$$(2.20)\quad \ddot{\underline{a}} + T\,\ddot{\underline{R}}' + 2\dot{T}\,\dot{\underline{R}}' + \ddot{T}\,\underline{R}' \equiv$$

$$- a_1\omega^2 \cos\mu \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} + TT_* \begin{bmatrix} \ddot{R} - R\dot{E}^2 - R\dot{A}^2 \cos^2 E \\ R\ddot{E} + 2\dot{R}\dot{E} + R\dot{A}^2 \sin E \cos E \\ R\ddot{A} \cos E + 2\dot{R}\dot{A} \cos E - 2R\dot{A}\dot{E} \sin E \end{bmatrix} +$$

$$+2T\omega \begin{bmatrix} 0 & -\sin\mu & \cos\mu \\ \sin\mu & 0 & 0 \\ -\cos\mu & 0 & 0 \end{bmatrix} T_* \begin{bmatrix} \dot{R} \\ R\dot{E} \\ R\dot{A} \cos E \end{bmatrix} +$$

$$- T\omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\mu & -\sin\mu\cos\mu \\ 0 & -\sin\mu\cos\mu & \cos^2\mu \end{bmatrix} T_* \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} .$$

Noting that by orthogonality $T_*^t = T_*^{-1}$ and $T^t = T^{-1}$, we can extract the common matrix factor TT_* and write (2.20) in the form

$$(2.21)\quad \ddot{\underline{a}} + T\,\ddot{\underline{R}}' + 2\dot{T}\,\dot{\underline{R}}' + \ddot{T}\,\underline{R}' \equiv$$

$$TT_* \left\{ \begin{bmatrix} \ddot{R} - R\dot{E}^2 - R\dot{A}^2 \cos^2 E \\ R\ddot{E} + 2\dot{R}\dot{E} + R\dot{A}^2 \sin E \cos E \\ R\ddot{A} \cos E + 2\dot{R}\dot{A} \cos E - 2R\dot{A}\dot{E} \sin E \end{bmatrix} + \right.$$

$$+ 2\omega\, T_*^t \begin{bmatrix} 0 & -\sin\mu & \cos\mu \\ \sin\mu & 0 & 0 \\ -\cos\mu & 0 & 0 \end{bmatrix} T_* \begin{bmatrix} \dot{R} \\ R\dot{E} \\ R\dot{A} \cos E \end{bmatrix} +$$

$$- \omega^2 \; T_*^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\mu & -\sin\mu\cos\mu \\ 0 & -\sin\mu\cos\mu & \cos^2\mu \end{bmatrix} T_* \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} +$$

$$\left. - a_1 \omega^2 \cos\mu \; T_*^t \, T^t \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \right\}.$$

Performing the indicated matrix multiplications and using some trigonometric identities in the third term, we can verify that

(2.22) $\ddot{\underline{a}} + T \underline{\ddot{R}}' + 2\dot{T} \underline{\dot{R}}' + \ddot{T} \underline{R}' \equiv$

$$TT_* \left\{ \begin{bmatrix} \ddot{R} - R\dot{E}^2 - R\dot{A}^2 \cos^2 E \\ R\ddot{E} + 2\dot{R}\dot{E} + R\dot{A}^2 \sin E \cos E \\ R\ddot{A} \cos E + 2\dot{R}\dot{A} \cos E - 2 R\dot{A}\dot{E} \sin E \end{bmatrix} + \right.$$

$$+ 2\omega \begin{bmatrix} R\dot{E} \cos\mu \sin A + R\dot{A} \cos E (\sin\mu \cos E - \cos\mu \sin E \cos A) \\ -\dot{R} \cos\mu \sin A - R\dot{A} \cos E (\sin\mu \sin E + \cos\mu \cos E \cos A) \\ \dot{R}(\cos\mu \sin E \cos A - \sin\mu \cos E) + R\dot{E}(\sin\mu \sin E + \cos\mu \cos E \cos A) \end{bmatrix} +$$

$$-\omega^2 R \begin{bmatrix} 1 - (\cos\mu \cos E \cos A + \sin\mu \sin E)^2 \\ (\sin\mu \sin E + \cos\mu \cos E \cos A)(\cos\mu \sin E \cos A - \sin\mu \cos E) \\ \cos\mu \sin A (\cos\mu \cos E \cos A + \sin\mu \sin E) \end{bmatrix} +$$

$$\left. - a_1 \omega^2 \cos\mu \begin{bmatrix} \cos\mu (\cos\mu \sin E - \sin\mu \cos E \cos A) \\ \sin\mu \sin E \cos A + \cos\mu \cos E \\ \sin\mu \sin A \end{bmatrix} \right\}.$$

4. Forces

We turn now to a consideration of the specific force on the right side of (2.14). We shall include two forces, gravity and air drag. The gravitational specific force upon a body at O in Figure 3 can be expressed in terms of an infinite series. Explicity writing terms only out to the first zonal harmonic (J_2), this series becomes

$$(2.23) \quad \underline{F}/m = -\frac{G_M}{r^3}\,\underline{r}\left[1+\frac{3}{2}\,\frac{J_2}{(r/a_e)^2}\,(1-5\cos^2\varphi_*)+\ldots\right] + -\frac{G_M}{r^2}\cos\varphi_*\left[3\,\frac{J_2}{(r/a_e)^2}+\ldots\right]\underline{u}_3 .$$

Here $\underline{r}$ is the body position vector from earth's center, r is its magnitude, and φ_* is the angle it subtends with the polar-axis. G_M and J_2 are gravitational constants and $\underline{u}_3$ is a unit vector in the north polar direction. In using (2.23) further, we shall retain only the terms shown. Applying (2.4), (2.9), (2.13) and writing out the vector components in (2.23) in the inertial system, we obtain

$$(2.24) \quad \underline{F}_g/m = -\frac{G_M}{r^3}\left[1+\frac{3}{2}\,\frac{J_2}{(r/a_e)^2}\,(1-5\cos^2\varphi_*)\right]\left\{\begin{bmatrix} a_1 \cos\mu & \cos\theta \\ a_1 \cos\mu & \sin\theta \\ a_2 \sin\mu & \end{bmatrix} + TT_*\begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}\right\} - \frac{G_M}{r^2}\cos\varphi_*\; 3\left(\frac{J_2}{(r/a_e)^2}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Finally, to bring out a uniform TT_* factor on the left side, we note that $TT_*\,T_*^t\,T^t$ is a unit matrix and terms lacking this factor need to be multiplied by $T_*^t\,T^t$. Thus (2.24) can be written in the form

$$(2.25)\quad \underline{F}_g/m = -\frac{G_M}{r^3}\left[1+\frac{3}{2}\,\frac{J_2}{(r/a_e)^2}\,(1-5\cos^2\varphi_*)\right]$$

$$\cdot\, TT_*\left\{T_*^t\begin{bmatrix}0\\(a_2-a_1)\sin\mu\cos\mu\\a_1\cos^2\mu+a_2\sin^2\mu\end{bmatrix}+\begin{bmatrix}R\\0\\0\end{bmatrix}\right\}+$$

$$-\frac{G_M}{r^2}\cos\varphi_*\left(3\,\frac{J_2}{(r/a_e)^2}\right)TT_*\left\{T_*^t\begin{bmatrix}0\\\cos\mu\\\sin\mu\end{bmatrix}\right\}.$$

Now introduce the abbreviations

$$(2.26)\quad c_1 = a_1\cos^2\mu + a_2\sin^2\mu$$

$$c_2 = (a_1-a_2)\sin\mu\cos\mu$$

and perform the T_*^t multiplications to obtain

$$(2.27)\quad \underline{F}_g/m = -\frac{G_M}{r^3}\left[1+\frac{3}{2}\,\frac{J_2}{(r/a_e)^2}\,(1-5\cos^2\varphi_*)\right]$$

$$\cdot\, TT_*\begin{bmatrix}R+c_1\sin E - c_2\cos E\cos A\\c_1\cos E + c_2\sin E\cos A\\c_2\sin A\end{bmatrix}+$$

$$-\frac{G_M}{r^2}\cos\varphi_*\,\frac{3J_2}{(r/a_e)^2}\,TT_*\begin{bmatrix}\cos\mu\cos E\cos A+\sin\mu\sin E\\\sin\mu\cos E-\cos\mu\sin E\cos A\\-\cos\mu\sin A\end{bmatrix}.$$

The use of this result requires an evaluation of r and cos φ_*. We note from (2.12) and (2.13) that $\underline{r} = \underline{a} + \underline{R} = \underline{a} + T\,\underline{R}'$ so that

$$(2.28)\quad \underline{r}^2 = \underline{r}^t\underline{r}$$

$$= a^2 + R^2 + 2\underline{a}^t\,T\,\underline{R}'$$

$$= a^2 + R^2 + 2R\,[0,\ -c_2,\ c_1]\begin{bmatrix}\cos E\sin A\\ \cos E\cos A\\ \sin E\end{bmatrix}$$

$$= a^2 + R^2 + 2R\,(c_1\sin E - c_2\cos E\cos A)\ .$$

Also

$$(2.29)\quad \cos\varphi_* = \frac{\underline{r}^t\,\underline{u}_3}{r}$$

$$= \frac{1}{r}\,(\underline{a} + T\,\underline{R}')^t\,\underline{u}_3$$

$$= \frac{1}{r}\,[\,a_2\sin\mu + R\,(\cos\mu\cos E\cos A + \sin\mu\sin E)]$$

The drag specific force on a body at altitude D above the earth, traveling with velocity $\underline{V}$ relative to the earth is given

$$(2.30)\quad \underline{F}_d/m = -\frac{1}{2}\,g\,\rho(D)\,\alpha\,V\underline{V}$$

where $g\rho(D)$ is an empirical function, the air density in weight units at altitude D, and α is a drag coefficient which characterizes the body. Since $\underline{\dot{R}}'$ is the relative velocity vector expressed in the earth fixed u' system, we can write $\underline{V}$ in the inertial system as

$$(2.31)\quad \underline{V} = T\,\underline{\dot{R}}' = TT_*\begin{bmatrix}\dot{R}\\ R\dot{E}\\ R\dot{A}\cos E\end{bmatrix}$$

where the last expression is obtained from (2.18). Then also

$$(2.32)\qquad V^2 = \underline{\dot{R}}'^t \underline{\dot{R}}' = \dot{R}^2 + R^2\dot{E}^2 + R^2\dot{A}^2 \cos^2 E \quad .$$

Thus (2.30) becomes

$$(2.33)\qquad \underline{F}_d/m = -\frac{1}{2} g\rho(D)\,\alpha\, V\, T T_* \begin{bmatrix} \dot{R} \\ R\dot{E} \\ R\dot{A}\cos E \end{bmatrix}$$

The altitude D may be approximated very nearly by using the construction shown in Figure 5. This figure is analogous to Figure 2 but

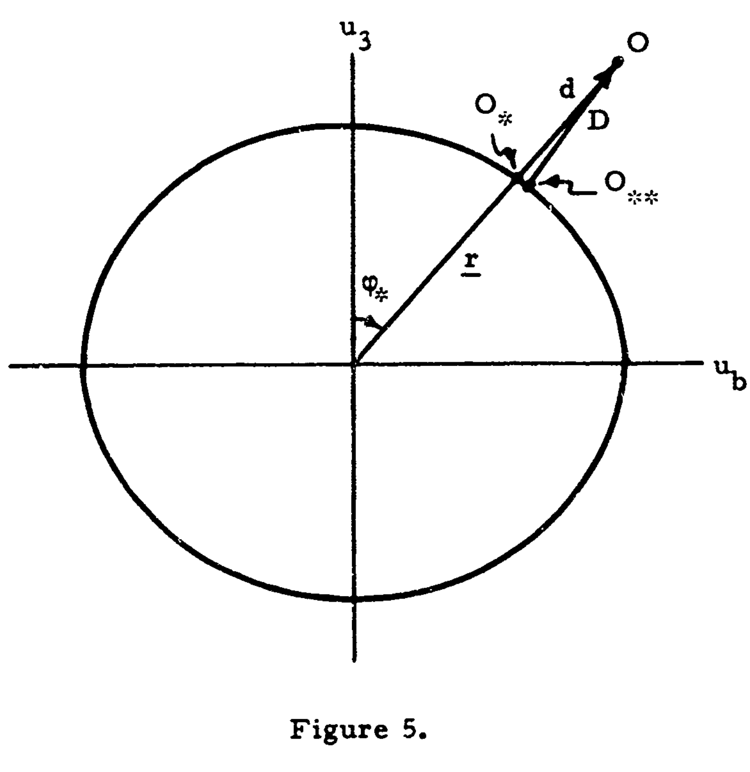

Figure 5.

is a cross-section of the earth in the plane of the object position vector $\underline{r}$ and polar axis $\underline{u}_3$. As before, O is the object's position while O_{**} is the foot of the perpendicular to the earth below. Segment OO_{**} is the true altitude. Instead, we compute and use the intercept OO_{**} of length d as

an approximation to D. This approximation is very good for small e. As with the similar expressions (2.1), the point O_* can be located in the parametric form

$$(2.34)\qquad \begin{aligned} u_b &= a_e \cos \beta_* \\ u_3 &= a_e \sqrt{1 - e^2}\, \sin \beta_* \quad . \end{aligned}$$

But also

$$(2.35)\qquad \begin{aligned} u_b &= (r-d) \sin \varphi_* \\ u_3 &= (r-d) \cos \varphi_* \quad . \end{aligned}$$

Eliminating β_*, u_b, u_3 from (2.34), (2.35), we find

$$\begin{aligned} a_e^2 &= u_b^2 + \frac{u_3^2}{1 - e^2} \\ &= (r-d)^2 \left(\sin^2 \varphi_* + \frac{\cos^2 \varphi_*}{1 - e^2}\right) \end{aligned}$$

and, therefore,

$$(2.36)\qquad D \simeq d = r - \frac{a_e \sqrt{1 - e^2}}{\sqrt{1 - e^2 \sin^2 \varphi_*}}$$

5. State Model Equations

Expressions (2.27) and (2.33), when added, complete the right side of the differential equations of motion (2.14). We have already evaluated the left side in (2.22). Note that all these expressions have a common matrix factor TT_* which we can discard or eliminate by multiplying throughout by $T_*^t\, T^t$. This final step in effect transforms all vectors into the u'' system

which is an orthogonalized radar coordinate (R, E, A) system. Thus, the component differential dynamical equations for the motion of the reentry body in radar polar coordinates can be read off.

In order to conform to requirements of the Kalman model, that the dynamical equations be first order differential equations as in (1.1), we identify as dynamical variables both R, E, A and $\dot{R}$, $\dot{E}$, $\dot{A}$. Since we shall be interested in estimating the drag parameter α as well, it will also be included in the state vector $\underline{x}$. Thus, define

(2.37)
$$\underline{x} = \begin{bmatrix} R \\ A \\ E \\ \dot{R} \\ \dot{A} \\ \dot{E} \\ \alpha \end{bmatrix}$$

where the order of variables E, A has been interchanged to match a current convention. Then the first three component equations of state (1.1) are actually kinematical,

(2.38)
$$\frac{dR}{dt} = \dot{R}$$
$$\frac{dA}{dt} = \dot{A}$$
$$\frac{dE}{dt} = \dot{E} \quad ,$$

while the second three are obtained from the derived equations of motion as noted above:

(2.39) Range equation

$$\frac{d\dot{R}}{dt} = R\dot{E}^2 + R\dot{A}^2 \cos^2 E$$

$$- 2\omega R[\dot{E} \cos\mu \sin A + \dot{A}\cos E (\sin\mu\cos E - \cos\mu\sin E\cos A)] +$$

$$+ \omega^2 R [1 - (\cos\mu\cos E\cos A + \sin\mu\sin E)^2] +$$

$$+ a_1 \omega^2 \cos\mu (\cos\mu\sin E - \sin\mu\cos E\cos A) +$$

$$- \frac{G_M}{r^3} \left[1 + \frac{3}{2} J_2 \left(\frac{a_e}{r}\right)^2 (1-5 \cos^2 \varphi_*)\right] [R + c_1 \sin E - c_2 \cos E\cos A] +$$

$$- \frac{2G_M}{r^2} \left[\frac{3}{2} J_2 \left(\frac{a_e}{r}\right)^2 \cos \varphi_*\right] [\cos\mu \cos E\cos A + \sin\mu \sin E] +$$

$$- \frac{1}{2} g\rho (D) \alpha V \dot{R}$$

(2.40) Azimuth equation (divide by RcosE)

$$\frac{d\dot{A}}{dt} = - \frac{2\dot{R}\dot{A}}{R} + 2\dot{A}\dot{E} \tan E$$

$$- \frac{2\omega}{\cos E} \left[\frac{\dot{R}}{R} (\cos\mu\sin E\cos A - \sin\mu\cos E) + \dot{E} (\sin\mu\sin E + \cos\mu \cos E\cos A)\right]$$

$$+ \frac{\omega^2}{\cos E} \cos\mu\sin A (\sin\mu\sin E + \cos\mu\cos E\cos A)$$

$$+ \frac{a_1 \omega^2}{R\cos E} \sin\mu\cos\mu\sin A$$

$$- \frac{G_M}{r^3 R\cos E} \left[1 + \frac{3}{2} J_2 \left(\frac{a_e}{r}\right)^2 (1 - 5 \cos^2 \varphi_*)\right] c_2 \sin A$$

$$+ \frac{2G_M}{r^2 R\cos E} \cos\varphi_* \left[\frac{3}{2} J_2 \left(\frac{a_e}{r}\right)^2\right] \cos\mu\sin A$$

$$- \frac{1}{2} g\rho (D) \alpha V \dot{A}$$

(2.41) Elevation equation (Divide by R)

$$\frac{d\dot{E}}{dt} = -\frac{2\dot{R}\dot{E}}{R} - \dot{A}^2 \sin E\cos E +$$

$$+ 2\omega\left[\frac{\dot{R}}{R}\cos\mu\sin A + \dot{A}\cos E\,(\sin\mu\sin E + \cos\mu\cos E\cos A)\right] +$$

$$+ \omega^2 (\sin\mu\sin E + \cos\mu\cos E\cos A)(\cos\mu\sin E\cos A - \sin\mu\cos E) +$$

$$+ \frac{a_1}{R}\,\omega^2 \cos\mu(\sin\mu\sin E\cos A + \cos\mu\cos E) +$$

$$- \frac{G_M}{r^3 R}\left[1 + \frac{3}{2} J_2\left(\frac{a_e}{r}\right)^2 (1-5\cos^2\varphi_*)\right]\left[c_1\cos E + c_2\sin E\cos A\right] +$$

$$- \frac{2G_M}{r^2 R}\cos\varphi_*\left[\frac{3}{2} J_2\left(\frac{a_e}{r}\right)^2\right]\left[\sin\mu\cos E - \cos\mu\sin E\cos A\right] +$$

$$- \frac{1}{2}\, g\,\rho(D)\,\alpha\, V\,\dot{E} \quad .$$

As a final component equation of the system model (1.1), we append the trivial relation

(2.42) $$\frac{d\alpha}{dt} = 0 \quad .$$

Thus for the present we assume the drag parameter to be constant.

6. Observation Model Equations

The next step in constructing a model for recursive trajectory estimation is to delineate the form of the observations (1.2). Since our dynamical description is in an $R, E, A, \dot{R}, \dot{E}, \dot{A}$ system, which is directly the set of observed quantities, the relations (1.2) have a very simple form. To be specific, let us suppose that at each observation time we measure

$R, A, E, \dot{R}$. Then the observation equation components are

(2.43)
$$
\begin{aligned}
z_1 &= R + w_1 \\
z_2 &= A + w_2 \\
z_3 &= E + w_3 \\
z_4 &= \dot{R} + w_4
\end{aligned}
$$

where w_1, w_2, w_3, and w_4 are the respective additive noises in the measurements with a known covariance matrix W.

We have noted previously that statistical calculations, the actual process of track smoothing and prediction, require a linearization of both the dynamical equations (2.38) through (2.42) and the observation equations (2.43). But the latter are already in linear form; the H matrix (1.12) for this set of measurements then has the simple form,

(2.44)
$$
H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}
$$

III. SPHERICAL EARTH CASE - JACOBIAN MATRIX

1. Simplified Equations of Motion

While the observation matrix H is trivial, the J matrix (1.8) for our dynamical system is very complicated. We have described in section I3 the use of the J matrix in the tracking procedure. It is used in (1.14) to extrapolate the Σ matrix between data times while the dynamical equations (1.13) are used to extrapolate the estimated trajectory. Essentially, then, the J matrix is only used in error propagation, and hence need not be calculated numerically with the same accuracy as the $\underline{f}$ functions. In fact, the only effect of slightly inaccurate J's is to produce slightly inaccurate Σ's which result in less than optimal estimates, $\hat{\underline{x}}$'s. The effect is analogous to the result of perturbing the value of a variable y near a minimum value of a function g(y). Relatively large excursions of y are often tolerable before the value of g(y) is appreciably raised.

It is very convenient for our problem to simplify J. For this reason, and with the above justification, we first reduce our $\underline{f}$ functions (2.39), (2.40), (2.41) to the case of a spherical earth, (e = 0), and drop the J_2 gravitational terms, (J_2 = 0). These simplifications also imply that

(3.1)
$$a_1 = a_2 = a = c_1$$

$$c_2 = 0$$

$$r^2 = a^2 + R^2 + 2aR\sin E$$

$$\cos\varphi_* = \frac{1}{r}\left[a\sin\mu + R(\cos\mu\cos E\cos A + \sin\mu\sin E)\right]$$

$$D = r - a + h_s \quad .$$

Making the above simplifications we note that (2.38) remains unchanged

$$(3.2)\qquad \frac{dR}{dt} = \dot{R}$$

$$\frac{dA}{dt} = \dot{A}$$

$$\frac{dE}{dt} = \dot{E} \ .$$

while (2.39), (2.40), (2.41), respectively simplify to:

$$\begin{aligned}(3.3)\qquad \frac{d\dot{R}}{dt} = {} & R\dot{E}^2 + R\dot{A}^2\cos^2 E \, + \\ & - 2\omega R\Big[\dot{E}\cos\mu\sin A + \dot{A}\cos E(\sin\mu\cos E - \cos\mu\sin E\cos A)\Big] + \\ & + \omega^2 R\Big[1 - (\cos\mu\cos E\cos A + \sin\mu\sin E)^2\Big] + \\ & + a\omega^2\cos\mu(\cos\mu\sin E - \sin\mu\cos E\cos A) + \\ & - \frac{G_M}{r^3}(R + a\sin E) - \frac{1}{2}\, g\rho(D)\alpha V\dot{R} \ .\end{aligned}$$

$$\begin{aligned}(3.4)\qquad \frac{d\dot{A}}{dt} = {} & - \frac{2\dot{R}\dot{A}}{R} + 2\dot{A}\dot{E}\tan E \, + \\ & - \frac{2\omega}{\cos E}\Big[\frac{\dot{R}}{R}(\cos\mu\sin E\cos A - \sin\mu\cos E) + \\ & \qquad + \dot{E}(\sin\mu\sin E + \cos\mu\cos E\cos A)\Big] + \\ & + \frac{\omega^2}{\cos E}\cos\mu\sin A(\sin\mu\sin E + \cos\mu\cos E\cos A) + \\ & + \frac{a\omega^2}{R\cos E}\sin\mu\cos\mu\sin A \, + \\ & - \frac{1}{2} g\rho(D)\alpha V\dot{A} \ .\end{aligned}$$

$$(3.5)\qquad \frac{d\dot{E}}{dt} = - \frac{2\dot{R}\dot{E}}{R} - \dot{A}^2 \sin E \cos E \; +$$

$$+ 2\omega\left[\frac{\dot{R}}{R} \cos\mu \sin A + \dot{A} \cos E(\sin\mu \sin E + \cos\mu \cos E \cos A)\right] +$$

$$+ \omega^2(\sin\mu \sin E + \cos\mu \cos E \cos A)(\cos\mu \sin E \cos A - \sin\mu \cos E) +$$

$$+ \frac{a}{R}\omega^2 \cos\mu(\sin\mu \sin E \cos A + \cos\mu \cos E) +$$

$$- \frac{G_M}{r^3 R} a \cos E - \frac{1}{2} g\rho(D)\alpha V\dot{E} \quad .$$

and again

$$(3.6)\qquad \frac{d\alpha}{dt} = 0 \; .$$

2. Jacobian Matrix

The J matrix has the following form (whether simplified or not).

$$(3.7)\qquad J \equiv \frac{\partial(f_R, f_A, f_E, f_{\dot{R}}, f_{\dot{A}}, f_{\dot{E}}, f_\alpha)}{\partial(R, A, E, \dot{R}, \dot{A}, \dot{E}, \alpha)}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ J_{\dot{R}R} & J_{\dot{R}A} & J_{\dot{R}E} & J_{\dot{R}\dot{R}} & J_{\dot{R}\dot{A}} & J_{\dot{R}\dot{E}} & J_{\dot{R}\alpha} \\ J_{\dot{A}R} & J_{\dot{A}A} & J_{\dot{A}E} & J_{\dot{A}\dot{R}} & J_{\dot{A}\dot{A}} & J_{\dot{A}\dot{E}} & J_{\dot{A}\alpha} \\ J_{\dot{E}R} & J_{\dot{E}A} & J_{\dot{E}E} & J_{\dot{E}\dot{R}} & J_{\dot{E}\dot{A}} & J_{\dot{E}\dot{E}} & J_{\dot{E}\alpha} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Performing the indicated differentions with the right side of (3.2) through (3.6) we can derive the individual J components. We note the expression for V in (2.32); we also use the fact that for any state variable, x,

$$(3.8)\qquad \frac{\partial\rho}{\partial x} = \frac{d\rho}{dD}\cdot\frac{\partial D}{\partial x}$$

$$= \rho'(D)\frac{\partial D}{\partial x}\ .$$

The individual simplified, spherical earth J matrix components are

$$(3.9)\qquad J_{\dot{R}R} = \dot{E}^2 + \dot{A}^2\cos^2 E +$$

$$- 2\omega\left[\dot{E}\cos\mu\sin A + \dot{A}\cos E(\sin\mu\cos E - \cos\mu\sin E\cos A)\right] +$$

$$+ \omega^2\left[1 - (\cos\mu\cos E\cos A + \sin\mu\sin E)^2\right] +$$

$$- \frac{G_M}{r^3} + \frac{3G_M}{r^5}(R + a\sin E)^2 +$$

$$- \frac{1}{2}g\rho(D)\alpha\dot{R}R\,\frac{\dot{E}^2 + \dot{A}^2\cos^2 E}{V} +$$

$$- \frac{1}{2}g\rho'(D)\alpha\dot{R}(R + a\sin E)\,\frac{V}{r}\ .$$

$$(3.10)\qquad J_{\dot{R}A} = - 2\omega R\cos\mu(\dot{E}\cos A + \dot{A}\sin E\cos E\sin A) +$$

$$+ 2\omega^2 R\cos\mu\cos E\sin A(\cos\mu\cos E\cos A + \sin\mu\sin E) +$$

$$+ a\omega^2\cos\mu\sin\mu\cos E\sin A\ .$$

$$(3.11)\qquad J_{\dot{R}E} = -2R\dot{A}^2\sin E\cos E +$$

$$+ 2\omega R\dot{A}\left[2\sin\mu\sin E\cos E + \cos\mu(\cos^2 E - \sin^2 E)\cos A\right] +$$

$$- 2\omega^2 R(\cos\mu\cos E\cos A + \sin\mu\sin E)(\sin\mu\cos E - \cos\mu\sin E\cos A) +$$

$$+ a\omega^2\cos\mu(\cos\mu\cos E + \sin\mu\sin E\cos A) +$$

$$- \frac{G_M a\cos E}{r^3} + \frac{3G_M}{r^5}aR\cos E(R + a\sin E) +$$

$$+\frac{1}{2}g\rho(D)\alpha\dot{R}\,\frac{R^2\dot{A}^2\sin E\cos E}{V}\ +$$

$$-\frac{1}{2}g\rho'(D)\alpha\dot{R}aR\cos E\,\frac{V}{r}\ .$$

(3.12) $$J_{\dot{R}\dot{R}} = -\frac{1}{2}g\rho(D)\alpha(V+\frac{\dot{R}^2}{V})\ .$$

(3.13) $$J_{\dot{R}\dot{A}} = 2R\dot{A}\cos^2E - 2\omega R\cos E(\sin\mu\cos E - \cos\mu\sin E\cos A)\ +$$

$$-\frac{1}{2}g\rho(D)\alpha\dot{R}\,\frac{R^2\dot{A}\cos^2E}{V}\ .$$

(3.14) $$J_{\dot{R}\dot{E}} = 2R\dot{E} - 2\omega R\cos\mu\sin A\ +$$

$$-\frac{1}{2}g\rho(D)\alpha\dot{R}\,\frac{R^2\dot{E}}{V}\ .$$

(3.15) $$J_{\dot{R}\alpha} = -\frac{1}{2}g\rho(D)\dot{R}V\ .$$

(3.16) $$J_{\dot{A}R} = \frac{2\dot{R}\dot{A}}{R^2} + \frac{2\omega\dot{R}}{R^2\cos E}(\cos\mu\sin E\cos A - \sin\mu\cos E)\ +$$

$$-\frac{a\omega^2}{R^2\cos E}\cos\mu\sin\mu\sin A\ +$$

$$-\frac{1}{2}g\rho(D)\alpha\dot{A}R\,\frac{\dot{E}^2+\dot{A}^2\cos^2E}{V}\ +$$

$$-\frac{1}{2}g\rho'(D)\alpha\dot{A}(R+a\sin E)\,\frac{V}{r}\ .$$

(3.17) $$J_{\dot{A}A} = \frac{2\omega}{\cos E}\cos\mu\sin A(\frac{\dot{R}}{R}\sin E + \dot{E}\cos E)\ +$$

$$+\frac{\omega^2}{\cos E}\cos\mu\left[\sin\mu\sin E\cos A + \cos\mu\cos E(\cos^2A - \sin^2A)\right]\ +$$

$$+\frac{a\omega^2}{R\cos E}\cos\mu\sin\mu\cos A\ .$$

(3.18) $$J_{\dot{A}E} = \frac{2\dot{A}\dot{E}}{\cos^2 E} - \frac{2\omega}{\cos^2 E}\left(\frac{\dot{R}}{R}\cos\mu\cos A + \dot{E}\sin\mu\right) +$$

$$+ \frac{\omega^2}{\cos^2 E}\cos\mu\sin\mu\sin A + \frac{a\omega^2}{R}\frac{\sin E}{\cos^2 E}\cos\mu\sin\mu\sin A \ +$$

$$+ \frac{1}{2}g\rho(D)\alpha\dot{A}^3R^2\frac{\sin E\cos E}{V} \ +$$

$$- \frac{1}{2}g\rho'(D)\alpha\dot{A}aR\cos E\frac{V}{r} \ .$$

(3.19) $$J_{\dot{A}\dot{R}} = -\frac{2\dot{A}}{R} - \frac{2\omega}{R\cos E}(\cos\mu\sin E\cos A - \sin\mu\cos E) \ +$$

$$- \frac{1}{2}g\rho(D)\alpha\dot{A}\frac{\dot{R}}{V} \ .$$

(3.20) $$J_{\dot{A}\dot{A}} = -\frac{2\dot{R}}{R} + 2\dot{E}\tan E - \frac{1}{2}g\rho(D)\alpha V \ +$$

$$- \frac{1}{2}g\rho(D)\alpha\frac{\dot{A}^2R^2\cos^2 E}{V} \ .$$

(3.21) $$J_{\dot{A}\dot{E}} = 2\dot{A}\tan E - \frac{2\omega}{\cos E}(\sin\mu\sin E + \cos\mu\cos E\cos A)$$

$$- \frac{1}{2}g\rho(D)\alpha\dot{A}\frac{R^2\dot{E}}{V} \ .$$

(3.22) $$J_{\dot{A}\alpha} = -\frac{1}{2}g\rho(D)\dot{A}V \ .$$

$$J_{\dot{E}R} = \frac{2\dot{R}\dot{E}}{R^2} - \frac{2\omega\dot{R}}{R^2}\cos\mu\sin A \; + \tag{3.23}$$

$$- \frac{a\omega^2}{R^2}\cos\mu(\sin\mu\sin E\cos A + \cos\mu\cos E) \; +$$

$$+ \frac{G_M a\cos E}{r^3\,R^2} + \frac{3G_M a\cos E(R + a\sin E)}{r^5 R} \; +$$

$$- \frac{1}{2}g\rho(D)\alpha\dot{E}R\,\frac{\dot{E}^2 + \dot{A}^2\cos^2 E}{V} \; +$$

$$- \frac{1}{2}g\rho'(D)\alpha\dot{E}(R + a\sin E)\frac{V}{r} \; .$$

$$J_{\dot{E}A} = 2\omega\cos\mu\left(\frac{\dot{R}}{R}\cos A - \dot{A}\cos^2 E\sin A\right) \; + \tag{3.24}$$

$$- \omega^2\cos\mu\sin A\left[2\cos\mu\sin E\cos E\cos A - \sin\mu(\cos^2 E - \sin^2 E)\right] +$$

$$- \frac{a}{R}\,\omega^2\cos\mu\sin\mu\sin E\sin A \; .$$

$$J_{\dot{E}E} = - \dot{A}^2(\cos^2 E - \sin^2 E) \; + \tag{3.25}$$

$$+ 2\omega\dot{A}\left[\sin\mu(\cos^2 E - \sin^2 E) - 2\cos\mu\sin E\cos E\cos A\right] \; +$$

$$+ \omega^2\Big[(\sin\mu\sin E + \cos\mu\cos E\cos A)^2 \; +$$

$$- (\sin\mu\cos E - \cos\mu\sin E\cos A)^2\Big] \; +$$

$$+ \frac{a}{R}\,\omega^2\cos\mu(\sin\mu\cos E\cos A - \cos\mu\sin E) \; +$$

$$+ \frac{G_M a\sin E}{r^3 R} + \frac{3G_M a^2\cos^2 E}{r^5} \; +$$

$$+ \frac{1}{2} g\rho(D)\alpha\dot{E}\,\frac{R^2\dot{A}^2 \sin E \cos E}{V} \; +$$

$$- \frac{1}{2} g\rho'(D)\alpha\dot{E} a R \cos E \frac{V}{r} \; .$$

(3.26) $$J_{\dot{E}\dot{R}} = -\frac{2\dot{E}}{R} + \frac{2\omega}{R}\cos\mu \sin A - \frac{1}{2} g\rho(D)\alpha\dot{E}\,\frac{\dot{R}}{V} \; .$$

(3.27) $$J_{\dot{E}\dot{A}} = -2\dot{A}\sin E \cos E + 2\omega \cos E(\sin\mu \sin E + \cos\mu \cos E \cos A) \; +$$

$$- \frac{1}{2} g\rho(D)\alpha\dot{E}\,\frac{R^2\dot{A}\cos^2 E}{V} \; .$$

(3.28) $$J_{\dot{E}\dot{E}} = -\frac{2\dot{R}}{R} - \frac{1}{2} g\rho(D) V - \frac{1}{2} g\rho(D)\alpha\,\frac{\dot{E}^2 R^2}{V} \; .$$

(3.29) $$J_{\dot{E}\alpha} = -\frac{1}{2} g\rho(D)\dot{E} V \; .$$

IV. STATISTICAL TESTING PROCEDURES

1. Basic Procedure

It is evident from the foregoing description that the equations for tracking in radar polar coordinates are relatively complex. Although one can organize the calculations to take advantage of common trignonometric expressions, etc., in the formulas, the program running time for such calculations considerably exceeds that for a simpler coordinate system (earth fixed rectangular) with a simpler model (spherical earth). In order to compare the radar polar coordinate tracking procedure with simpler versions of itself and with other procedures a systematic testing method is desirable.

Such a testing method can be implemented by a computer simulation which statistically compares the tracking runs made on a given trajectory (or set of trajectories) using the full model against similar runs made with a simplified or alternate model. Let us call the full model the general model and the various alternate models, special models.

The statistical comparisons can be implemented by the following procedure.

a) Select a given trajectory by specifying its initial conditions and the drag characteristics of the body. Select the radar site location. Specify statistical a priori error variances for the initial trajectory and drag (state) variables. Select the quantities to be measured, the data sampling rate, and the measurement error covariance matrix.

b) Pass 1. Simulate the trajectory of the reentry object by integrating the equations of motion as given by the general model with the initial conditions assumed above. This generated trajectory is assumed to be the true trajectory. Record its values $\underline{x}_*(n)$ (state vector) at the sampling times.

At the same time make and record $\underline{z}(n)$. These are measurements of the true trajectory with an added random noise vector chosen by a Monte Carlo process from a multidimensional normal distribution with the assumed covariance matrix. (Since the coordinate system of current concern is radar-centered polar, the H(n) matrices (2.44) are trivial and need not be specifically computed or stored. In general, however, $H_*(n)$ would be evaluated on the reference trajectory and recorded.)

Perform recursive tracking as outlined in section I using again the full model with the data $\underline{z}(n)$ and the above input parameters. Evaluate partial derivatives (the J matrix computation) at the true trajectory points rather than at the estimated trajectory values. This tracking procedure provides an ideal, optimum set of estimates $\hat{\underline{x}}_*(n|n-1)$, $\hat{\underline{x}}_*(n|n)$ and their associated covariances $\Sigma_*(n|n)$ which are recorded.

The recorded quantities provide a standard against which alternate, simplified tracking procedures may be compared.

c) Pass 2. Using pass 1 observations, $\underline{z}(n)$, as data, perform the tracking according to the special alternate model to be tested. This tracking generates the estimates $\hat{\underline{x}}(n|n-1)$, $\hat{\underline{x}}(n|n)$ and their covariances $\Sigma(n|n-1)$, $\Sigma(n|n)$. Here we do not assume that the true trajectory is known; the J matrix is computed along the estimated trajectory, also the initial track estimates and covariances need not be identical to those of pass 1. This is a simulation of tracking as it would actually be performed in the real world.

During the pass 2 tracking we compute a series of comparison statistics to be described below. These comparison statistics determine whether the special model tracking has been significantly degraded from the ideal.

2. Tests of Trajectory Fit

In order to derive these statistics it is convenient to view the Kalman tracking process as a very general method for least squares curve fitting.

For example, suppose that a curve of the form

(4.1) $$Ae^{-an} + Be^{-bn} + Ce^{-cn} \quad .$$

with parameters A, a, B, b, C, c is to be fitted to data points, z(n), n = 1, 2, 3, ... N. This problem can be linearized and formulated according to the Kalman model with the state vector

$$\underline{x} = \begin{bmatrix} A \\ a \\ B \\ b \\ C \\ c \end{bmatrix}$$

having the trivial dynamics $\frac{d\underline{x}}{dt} = 0$. The computational scheme of section I can be immediately adapted to this problem.

Suppose we are sure (the null hypothesis) that the data arise from a model of the above form; then we may ask the question whether the expression (4.1) for the curve can be simplified by dropping the third term, viz.,

(4.2) $$Ae^{-an} + Be^{-bn} \quad .$$

A rather obvious procedure for determining the suitability of the special expression (4.2) in representing the data is to perform a least squares fit of (4.2) to the data, z(n), and to examine the residuals

(4.3) $$z(n) - (Ae^{-an} + Be^{-bn})$$

after the best fit has been obtained. Under the null hypothesis the expected size of the residuals can be determined. If the residuals are large, in some collective sense, with respect to the expected magnitude of the discrepancy,

then the special model (4.2) is detectably poorer than the general model (4.1) in its ability to fit the data. The simplification has significantly degraded the model.

This inspection of the residuals can be used in a similar fashion to test alternatives to (4.1) even though they may have a completely different form. For instance, instead of (4.1) we might try a polynomial representation

$$(4.4) \qquad An^2 + Bn + C$$

to fit the data. Under some circumstances we might find that residuals with (4.4) are satisfactorily small and that, therefore, a polynomial curve will also fit the data. Note that such a type of residual test does not determine whether or not the best fit parameters $\hat{A}$, $\hat{B}$, $\hat{C}$ of (4.4) or $\hat{A}$, $\hat{a}$, $\hat{B}$, $\hat{b}$, or (4.2) are close to or correspond in any sense to the best fit parameters $\hat{A}$, $\hat{a}$, $\hat{B}$, $\hat{b}$, $\hat{C}$, c of the general model (4.1). Indeed, while the general model may have a physical basis for its form, the model (4.4) may be entirely empirical. Yet (4.4) may equally well fit the data; its estimates $\hat{A}$, $\hat{B}$, $\hat{C}$, then, are all that are needed to represent a fitted curve. In effect, they extract all of the available information from the curve data. The remainder, the residuals, appear typically random.

The above test of fit, then, is basically an answer to the question of whether or not simplifications or alterations in the curve fitting or, in general, the tracking model produce a loss of information during the process of estimation. A negative result indicates that the reduced model complexity is unaccompanied by the penalty of information loss.

In order to formalize the above ideas let us assume that our model has been linearized and that its statistics are gaussian (measurement noise, random driving terms in state equation, initial conditions). Then, given the model, one can write down the joint density distribution of the data $\underline{z}(1)$, $\underline{z}(2), \ldots, \underline{z}(N)$ given the initial state value $\underline{x}_0$. This distribution has

the joint normal form

$$p(\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(N) | \underline{x}_0) = Ce^{-\frac{1}{2}Q} \tag{4.5}$$

where C is a constant and Q is a quadratic form in the z's. By the chain rule of conditioning we can also write (4.5) in the form

$$\begin{aligned} p(\underline{z}(1), \underline{z}(2), \ldots, \underline{z}(N) | \underline{x}_0) &+ \\ = p(\underline{z}(1,| \underline{x}_0) p(\underline{z}(2) | \underline{z}(1), \underline{z}_0) p(\underline{z}(3) | \underline{z}(1), \underline{z}(2), \underline{x}_0) &+ \\ \ldots\ p(\underline{z}(N) | \underline{z}(1), \ldots, \underline{z}(N-1), \underline{x}_0) &+ \\ = Ce^{-\frac{1}{2}(Q_1 + Q_2 + \ldots + Q_N)} & . \end{aligned} \tag{4.6}$$

where Q_n is the quadratic form in the density distribution $p(\underline{z}(n) | \underline{z}(1), \ldots, \underline{z}(n-1), \underline{z}_0)$.

It is now simple to identify $\hat{\underline{z}}(n|n-1)$, the expected value of $\underline{z}(n)$ given the previous data and initial conditions. Since $\underline{z}(n) = H(n)\underline{x}(n) + \underline{w}(n)$,

$$\hat{\underline{z}}(n|n-1) = H(n)\hat{\underline{x}}(n|n-1) \quad , \tag{4.7}$$

for the noise $\underline{w}(n)$, being independent of previous data, has conditional expectation zero. Thus Q_n has the form

$$Q_n = \left[\underline{z}(n) - H(n)\hat{\underline{x}}(n|n-1)\right]^t S_n^{-1} \left[\underline{z}(n) - H(n)\hat{\underline{x}}(n|n-1)\right] \tag{4.8}$$

where S_n is the conditional covariance matrix of $\underline{z}(n)$.

By direct calculation we find

$$(4.9)\qquad S_n = \mathrm{cov}\left[\underline{z}(n) - H(n)\hat{\underline{x}}(n|n-1)\right]$$

$$= \mathrm{cov}\left[H(n)(\underline{x}(n) - \hat{\underline{x}}(n|n-1)) + \underline{w}(n)\right]$$

$$= H(n)\Sigma(n|n-1)H^t(n) + W(n) \quad ,$$

since $\underline{x}(n) - \hat{\underline{x}}(n|n-1)$ and $\underline{w}(n)$ are independent. Thus reviewing (4.5), (4.6), (4.8), (4.9) we see that the data distribution (4.5) which may also be termed the likelihood function uses the quadratic form

$$(4.10)\qquad Q = \sum_{n=1}^{N} (\underline{z}(n) - H(n)\hat{\underline{x}}(n|n-1))^t \left[H(n)\Sigma(n|n-1)H^t(n) + W(n)\right]^{-1} \cdot$$

$$\cdot (\underline{z}(n) - H(n)\hat{\underline{x}}(n|n-1)) \quad .$$

This quadratic form, being in the exponent of a joint normal distribution, has a chi-square distribution. It is evident that the vector difference $\underline{z}(n) - H(n)\hat{\underline{x}}(n|n-1)$ is a type of residual showing the deviation of the new data $\underline{z}(n)$ from its position expected on the basis of the past data and initial conditions. Thus expression (4.10) is a measure of data fit over the span $n = 1, 2, \ldots, N$. Large values of Q indicate bad fit, small values indicate good fit. Statisticians will also note that Q is -2 times the natural log of the likelihood function and that the value of this function indicates the degree to which the data conforms to the assumed model.

The dimensionality of Q, the number of degrees of freedom, is, on the face of it, Np, where each z vector has p components. This will be true in nondegenerate cases where the residuals in (4.10) and their associated covariances are not infinitely small or infinitely large and are consistently computed. The one important practical variation to this rule is the frequently useful, but inconsistent, assumption of an initial $\underline{x}_0$ vector having finite (small) components but initial $\Sigma(1|0)$ matrix with infinite (very large) variance components. This inconsistency manifests itself in the computation of Q where some of the initial residuals are much smaller than their purported

variance. Thus, no preceptible contribution to Q results. In order to correct for this effect, we must subtract from Np the number of $\underline{x}_0$ vector components so handled. The result is the correct number of degrees of freedom, ν, for Q. For example, in a tracking model where four observations are taken at each time step and $\underline{x}_0$ has seven components, the two extreme instances are as follows,

$$(4.11) \qquad \nu = \begin{cases} 4N & \text{initial } \underline{x},\ \Sigma \text{ consistent} \\ 4N - 7 & \text{initial } \underline{x} \text{ components all} \ll \text{than initial } \Sigma \text{ variances} \end{cases} .$$

3. Variations on the Test of Fit

The Q statistic may now be used in the two pass testing procedure mentioned previously. $\underline{z}(n)$ are generated on the 1st pass with the full model; $\hat{\underline{x}}(n|n-1)$ on the 2nd pass with the special model to be tested. We have two choices for the covariance matrices $\Sigma_*(n|n-1)$ from pass 1 or the estimated covariances on pass 2. The pass 1 covariances actually supply the required statistic because the pass 2 estimated covariances are subject to additional computational errors that the estimated trajectory must be substituted for the true trajectory in the linearization process. (In our current application H(n) is not affected by this distinction since in radar polar coordinates no linearization of the observational equations is required. In general, however, H(n) would be derived by a linearization and a corresponding choice of evaluation on the true trajectory, — from pass 1 — or on the estimated trajectory would be afforded.

For certain special purposes we might prefer, nevertheless, to use the estimated covariances to calculate Q. Our interest is due to the fact that the matrix $H\Sigma H^t + W$ and the residual vector $z - H\hat{x}$ occur in the estimate and covariance recursions (1.15), (1.16) of the tracking process as well as in the Q terms. Numerical difficulties can occur in the tracking which lead to instabilities if the estimated matrix $H\Sigma H^t + W$ departs significantly from the ideal. If such difficulties are present, the residual vector

will not compare well with $H\Sigma H^t + W$ and the Q term will tend to be large. Note that here the situation is essentially the reverse of the previous one where $H\Sigma H^t + W$ is known to be correct and the residuals are compared with it. In the present situation the accuracy of the estimated $H\Sigma H^t + W$ is also in question. For such a situation the Q statistic provides a useful measure of the discrepancy although no specific numerical probabilities can be attached to the measure because the Q sampling distribution is then not simply chi-square. By comparing the Q's obtained by using both ideal and estimated Σ's we can qualitatively assess the additional covariance estimate errors and stability properties of the tracking recursions.

We note from expression (4.10) that a Q value may be computed for any particular number of successive data points N. Strictly speaking, a numerically correct test of fit requires us to fix N, (e.g., to include all the data) before calculating Q since the successive increasing values of Q are not statistically independent. It is convenient, nevertheless, to monitor these successive values so as to get an approximate idea of the way the trajectory fit is progressing. In order to expedite this monitoring it is convenient to convert Q to an equivalent unit normally distributed variable, X. Thus we map the chi-squared distribution and regions shown in Figure 6a onto the unit normal distribution and regions in Figure 6b.

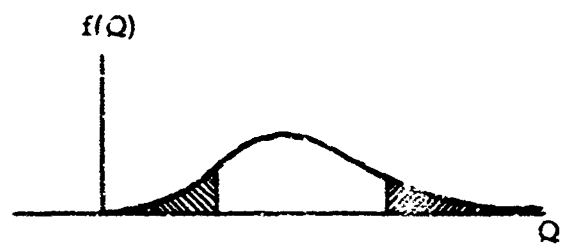

Figure 6a.

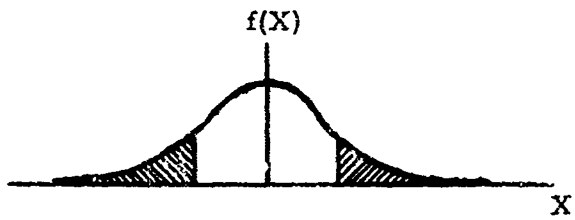

Figure 6b.

This mapping depends on the number of degrees of freedom and is given approximately by the formula [5]

$$X = \frac{(Q/\nu)^{1/3} - (1 - 2/(9\nu))}{\sqrt{2/(9\nu)}} \quad . \tag{4.12}$$

On the basis of chance we expect that X will vary roughly between -2 and +2. If $X > 2$, then the residuals are too large, the fit is bad and the model is suspect. If $X < -2$, the fit is too good; this circumstance is usually due to a programming error.

4. Test of Estimate Accuracy

Now suppose that the special model fit has been tested and is satisfactory. A further question is whether the estimates, $\hat{\underline{x}}(n|n)$, produced by the special model are close to the true trajectory values $\underline{x}_*(n)$. That this need not be the case is shown by our curve fitting example noted previously. The state variables describing the polynomial (4.4) of the special model are not directly comparable to the variables in the exponential expression (4.1) describing the general model; yet the fit can be excellent. In many instances, however, the intent of the model specialization will be not only to preserve the fit but also to preserve the significance of the model parameters. In these instances we hope to find close correspondence between $\underline{x}_*(n)$ components and all or some of the $\hat{\underline{x}}(n|n)$ components.

An appropriate test for this correspondence (for the entire vector) is to form the estimate error, $\hat{\underline{x}}(n|n) - \underline{x}_*(n)$, and compare it with its theoretical ideal covariance $\Sigma_*(n|n)$ in a statistic,

$$P = \left[\hat{\underline{x}}(n|n) - \underline{x}_*(n)\right]^t \Sigma_*^{-1}(n|n) \left[\hat{\underline{x}}(n|n) - \underline{x}_*(n)\right] \quad . \tag{4.13}$$

This statistic can be computed for each n and, except for the first few sample points where degeneracies may occur as noted earlier, it has a chi-square distribution with q degrees of freedom, where q is the dimension of $\underline{x}_*$.

If $\hat{\underline{x}}(n|n)$ is significantly distorted from the true $\underline{x}_*(n)$ the P statistic will be large. A typical situation for a two dimensional state vector as shown in Figure 7.

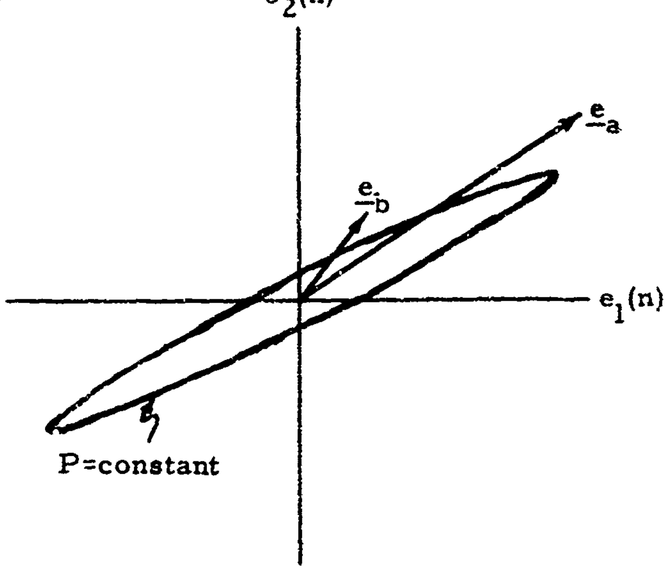

Figure 7.

We see an error ellipse of constant P and constant error probability as determined by $\Sigma_*(n|n)$. Let us assume that this ellipse marks the critical P level. If the error vector lies outside this ellipse the error is significantly large. Such an error vector $\underline{e}_a$, is shown in the figure.

However, note vector $\underline{e}_b$. This vector also is significantly large. But owing to the high statistical correlation between error components, causing a very skewed error ellipse, the individual components of $\underline{e}_b$, when tested one at a time against their variances, are not individually large. Thus it is possible for the combined error vector test to show up significant distortions even though individual components of error are well within their respective expected limits.

For this reason it is appropriate to examine specific components or subsets of the vector $\hat{\underline{x}}(n|n)$ for the desired correspondences based on the physical requirements of the problem. For example, in a trajectory tracking problem we might separate the trajectory position and velocity components

of $\hat{\underline{x}}$ into one subvector and the drag parameter components into another and test each separately against corresponding components of $\underline{x}_*$.

One further test variation is useful. Under some circumstances our special model parameters may be chosen to be specific functions of the general model parameters. Or we may be interested in a subsidiary estimation of such parameter functions although the recursive estimation does not produce them directly. For example, let us consider a vector y of functions of the general model state at time n.

$$(4.14) \qquad \underline{y}(n) = \underline{y}(\underline{x}_*(n)) \quad .$$

Suppose that our estimation process produces the estimates $\hat{\underline{y}}(n|n)$ by the calculation,

$$(4.15) \qquad \hat{\underline{y}}(n|n) = \underline{y}(\hat{\underline{x}}(n|n)) \quad .$$

Then the errors $\underline{y}(n) - \hat{\underline{y}}(n|n)$ may be compared with the theoretical covariance as before in a chi-square test. For this comparison we need

$$(4.16) \qquad \mathrm{cov}\left[\hat{\underline{y}}(n|n)\right] = \mathrm{cov}\left[\underline{y}(\hat{\underline{x}}(n|n)\right]$$

$$= \left[\frac{\partial \underline{y}}{\partial \underline{x}}\right] \mathrm{cov}(\hat{\underline{x}}(n|n)) \left[\frac{\partial \underline{y}}{\partial \underline{x}}\right]^t$$

or

$$(4.17) \qquad S(n) = \Gamma(n)\, \Sigma_*(n|n)\, \Gamma^t(n)$$

where $\Gamma(n)$ is the partial derivative matrix of the $\underline{y}$ functions with respect to the $\underline{x}$ variables evaluated along the true trajectory. If we assume no degeneracy, then the number of degrees of freedom for the chi-squared test statistic,

$$P = \left[\underline{y}(n) - \hat{\underline{y}}(n|n) \right]^t S^{-1}(n) \left[\underline{y}(n) - \hat{\underline{y}}(n|n) \right] \tag{4.17}$$

is equal to the number of $\underline{y}$ components.

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NOTATION AND SYMBOL LIST

General Notation

$(\ )^t$	matrix transpose
$(\ )^{-1}$	matrix inverse
$(\underline{\ })$	vector
$(\underline{\ })_i$	vector components
$(\ )_{ij}$	matrix components
$p(\ )$	probability density distribution
$\frac{d}{dt}(\ )$	time derivative, equation of state
$(\dot{\ })$	time derivative, dynamic equations of motion
$(\ )_0$	quantity evaluated on reference trajectory
$\delta(\ )$	deviation from the reference value
$\underline{x}(n)$	$\underline{x}$ (or other quantity) at time $t = t_r$
$cov(\ )$	covariance operator
$(\hat{\ })$	estimate
$\hat{\underline{x}}(n\|k)$, $\Sigma(n\|k)$	estimate of $\underline{x}(n)$ and covariance of estimate given data up to and including time t_k
$(\ )_*$	section IV notation for covariances, estimates and observations calculated on pass 1
A, a, B, b, C, c	general parameters in section IV discussion

ARCON

<u>Coordinate Systems (orthogonal)</u>

u_1, u_2, u_3	inertial unit vectors $\underline{u}_1, \underline{u}_2, \underline{u}_3$ (Figure 1)
u_1', u_2', u_3	radar centered, topographic unit vectors $\underline{u}_1', \underline{u}_2', \underline{u}_3'$ (Figure 3)
u_1'', u_2'', u_3''	radar centered, u_1 along line of sight unit vectors $\underline{u}_1'', \underline{u}_2'', \underline{u}_3''$ (Figure 4)

<u>Conventions</u>

Let $\underline{Y}$ be a physical vector. Then we express

$\underline{Y}$	its u_1, u_2, u_3 coordinates
$\underline{Y}'$	its u_1', u_2', u_3' coordinates
$\underline{Y}''$	its u_1'', u_2'', u_3'' coordinates

$Y = |\underline{Y}| = |\underline{Y}'| = |\underline{Y}''|$

<u>Transformations</u>

$\underline{Y}' = T_* \underline{Y}''$

$\underline{Y} = T \underline{Y}'$

<u>Symbols</u>

$\underline{a}$, a	radar station vector, $a = |\underline{a}|$
a_1, a_2	station parameters

Symbols (cont.)

a_c	equatorial earth radius
A	radar azimuth coordinate
α	drag parameter
β, ϑ_*	auxiliary angles in ellipse geometry
c_1, c_2	derived quantities dependent on station parameters
C	constant in probability distribution
D, d	altitude of tracked object above earth, approximation to D
e	eccentricity of earth
e_1, e_2, $\underline{e}_a$, $\underline{e}_b$	estimate error components and vectors
E	radar elevation coordinate
$\underline{f}$	functions on right side of equations of state
$\underline{F}$, $\underline{F}_g$, $\underline{F}_d$	vector force, gravity, drag
G_M	principal gravitational constant
$\underline{\gamma}$	arbitrary functions of state variables
Γ	partial derivative matrix of $\underline{\gamma}$, $\Gamma_{ij} = \frac{\partial \gamma_i}{\partial x_j}$
$\underline{h}$	observation functions

Symbols (cont.)

h_s	station height
J	jacobian matrix, $J_{ij} = \frac{\partial f_i}{\partial x_j}$
J_2	first zonal harmonic gravitational constant
k	auxiliary constant
m	mass of tracked object
μ, μ'	geodetic, geocentric latitude
n	index denoting t_n
N	total number of observations
ν	number of degrees of freedom for chi-square
O, O_*, O_{**}	position points
p	dimension of $\underline{z}$
φ, φ_*	geocentric colatitude; station, object
q	dimension of $\underline{x}$
Q	quadratic form, test of fit
$\underline{r}$	object vector from earth center
$\underline{R}$, R	object vector from radar, radar range coordinate

Symbols (cont.)

$g\rho$ (D)	air density (weight units) at altitude D
S	covariance of $\underline{z}$
Σ, Σ_*	covariance of $\underline{x}$ estimates
t	time
T, T_*	transformation (rotation) matrices
θ	station longitude
$\underline{u}$, u	(with subscripts and primes) various coordinate system unit vectors and components
$\underline{V}$, V	velocity, speed of object relative to earth
w	observation noise
W	observation noise covariance
w_*	state equation driving term
W_*	driving term covariance
$\underline{x}$	state vector
$\underline{X}$	normal variate equivalent to chi-square variate
$\underline{y}$	variables equal to functions of $\underline{x}$
$\underline{z}$	observation vector
ω	earth rotation rate (siderial)

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13. ABSTRACT

The following report describes (a) an outline of the Kalman-based tracking procedure for reentry trajectories, (b) a detailed derivation of a mathematical model to implement this procedure in radar polar coordinates, and (c) methods for checking, via simulation, the degree of degradation introduced by model simplifications and other changes.

14. KEY WORDS

reentry physics	radar polar coordinate model
recursive tracking	ellipsoidal earth
Jacobian matrix	spherical earth

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